

# Intercities traffic flow: linear and nonlinear models

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We simulate the traffic of vehicles circulating within a network formed by sites (cities, car-rental agencies, parking lots, etc.) connected by two-way roads or highways, to which we shall refer generically as intercity traffic. Our aim is to forecast the flux of the vehicles for  $n$  consecutive days, by taking as prior knowledge previous observations and measurements. The formal tools used in our analysis consists in: (1) making use of digraphs that allow the visualization and schematization of the problem, where edges correspond to roads, streets, highways, etc. and the vertices with loops represent the sites. (2) From an initial set of transition probabilities, that form a Markov chain, or from an empirical distribution of vehicles at the vertices and edges, we construct the Markov matrix which is a stochastic matrix (SM), and assume that the evolution of the traffic flows according to the power  $n$  of the SM and, (3) the use of the Perron-Frobenius theory (for matrices having non-negative entries) to analyze the outcomes. We identify several modes or regimes before the distribution of vehicles attains its stationary state, and we calculate the characteristic decay times of the transient regimes. We analyze: (1) a closed four-site network model, (2) to this network we add an input and an output flux of vehicles; asymptotically ( $n \rightarrow \infty$ ), each  $n$ -step Markov matrix goes to a unique stationary one. And (3), we study the traffic flow ruled by a nonlinear evolution in time; we find out that this model permits the existence of several ( $L$ ) stationary states for the traffic flux. We determine that each stationary matrix should allow to forecast the state of the traffic for each one of chosen  $L$  different moments of the day, and that, asymptotically, the flux of vehicles is allowed to change with time cyclically, going from one stationary state to the other. The formulation of the proposed traffic models presented in this paper precede an empirical data analysis that is being currently carried out.

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## I. INTRODUCTION

The understanding of the mechanism and systematization of the flow of vehicles along roads and highways is a crucial point in the planning of mobility and construction of connections between sites. Mathematical models allied with observation have been proposed since the 1940-50 decades, see for instance the seminal papers [1, 2] and references therein. The need for forecasting the traffic behavior is necessary and is a challenging task for urban planners. For that aim the observation and the measurement of the traffic flow are essential operational practices that have to go along with theoretical approaches, mathematical modeling and numerical simulations in order to acquire a grasp of the problem of the traffic flow and jam, in order to hint for solutions and proposals of public policies for urban and intercity vehicular mobility. Researches have been developed in several places uninterruptedly and many articles and reports were published on the subject, among which we cite, for instance, [3–29], and in the books [30, 31].

The success of a mathematical model to emulate a complex vehicular traffic network depends essentially on its capability to describe and forecast the flux of the vehicles along the arteries, and these qualities depend on the choice of the adopted formalism. There are approaches that make use of fluid dynamics and differential equations [4, 7] whereas others follow the Markov chain and Perron-Frobenius theories [31–33].

Here we shall consider that the circulation of vehicles occurs essentially intercities, intracities, or intersites; by sites we mean towns, parking spaces or car-rental agencies. Within a network the vehicles do not circulate continuously – as it occurs with a fluid that flows through a net of pipes – but have to stop at lights, in garages, in parking lots, etc. Thus, traffic rules and allowed maximum speeds in the lanes are established in order to design an optimal flow of vehicles, by combining high mobility with minimal jam and crashes.

In this study our approach is not microscopic, namely we do not describe the characteristics of the vehicles, as mean

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size, mean speed, stopping distance nor the characteristics of the arteries as length, width and number of lanes. It consists in schematizing a traffic network imaged by a digraph, as shown in Fig. 1, that could be scaled up to higher dimensions and complexity. The mathematical elements we utilize are essentially the digraph theory, the Markov chain formalism and the Perron-Frobenius theory of matrices. We consider firstly two linear modeled scenarios: (1) a network consisting of four sites, each one connected to all the others by two-way roads or highway. The number of vehicles circulating in the roads plus those in the cities is a conserved quantity in time. (2) In the former network we insert, in addition, an input of new vehicles and an output of useless old ones, again at a daily basis (see the digraph in Fig. 4). To a digraph one associates a square matrix,  $\mathbb{A}$ , containing in its entries the information obtained empirically, namely, the average number of vehicles established by previous observations. To forecast the distribution of the vehicles in the network we construct a stochastic matrix,  $\mathbb{M}$ , obtained from  $\mathbb{A}$ , whose entries are proportional to the fractions of vehicles in sites and roads. In this way, the predictions will be probabilistic and we assume that the daily change of the distribution obeys an evolution law based on an  $n$ -step Markov process –  $n$  standing for the discretized time. This approach constitutes an adaptation of a model proposed in Ref. [34] for human migrations. We present the theory, work out illustrative numerical examples and analyze the results.

We extend our analysis by examining a third model, now a nonlinear  $n$ -step Markov process; by nonlinearity we mean that the entries of the stochastic matrix also depend on  $n$ . With a specific choice of periodic functions for the entries we verify the existence of several stationary regimes (instead of a single one as it occurs in the linear models), such that the vehicle fluxes change cyclically. This nonlinear approach enables more detailed predictions about the traffic flow because it permits to slice, for instance, a 24 hours interval of time into several subperiods of traffic observation in the stationary regimes. The present study is a theoretical approach to describe and forecast the traffic flow in a neighborhood of Tigre, a county the province of Buenos Aires, Argentina, using measurements that are currently in course.

## II. MODEL I: INTERCITY CIRCULATION OF VEHICLES BETWEEN SITES WITHOUT INPUT OR OUTPUT.

The network model consists of four sites to which we attribute the letters  $A, B, C, D$ , and a system of arteries (roads, highways, etc.). The traffic system contains a definite and conserved quantity of vehicles; in short, we have a conservative network. Pictorially we represent the network by a digraph, as shown in Fig. 1,

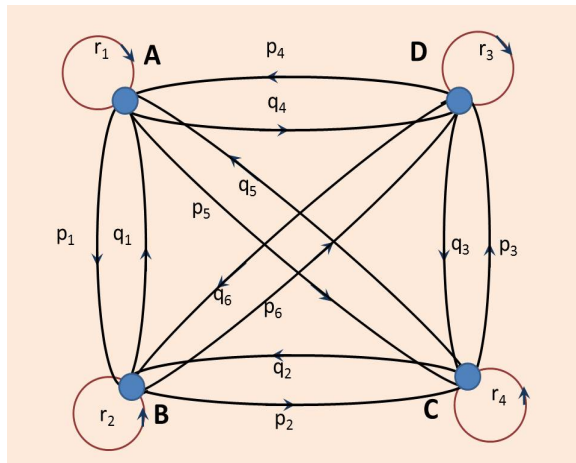


FIG. 1: Traffic flow network. Sites (cities or parking lots) are represented by the vertices  $A, B, C, D$  with respective loops; the edges represent the routes that connect the vertices. The letters  $r_i$ , for the loops, represent the fractions of the vehicles present in the sites whereas  $p_i$  and  $q_i$ , at the edges, stand for the fractions of vehicles circulating in the connecting arteries.

where each *loop*, associated to a *vertex*, corresponds to a site that accommodates a certain number of vehicles, whereas each *edge* is a link between the vertices that contains a number of vehicles in transit from one site to the other with directions represented by the arrows, as shown in Fig. 1.

To the digraph of Fig. 1 one associates the array shown in Table I, the sum of the entries of each row,  $U_i$ , and the sum of the entries of each column,  $T_i$ , are represented in Table I and in the last row one finds the sums representing the total number of vehicles  $Y$  present in the network.

–	$A$	$B$	$C$	$D$	Sum of the lines entries
$A$	$R_1$	$P_1$	$P_5$	$Q_4$	$U_1$
$B$	$Q_1$	$R_2$	$P_2$	$P_6$	$U_2$
$C$	$Q_5$	$Q_2$	$R_3$	$P_3$	$U_3$
$D$	$P_4$	$Q_6$	$Q_3$	$R_4$	$U_4$
Sum of the columns entries	$T_1$	$T_2$	$T_3$	$T_4$	–
–	$Y = \sum_{i=1}^4 T_i$				$Y = \sum_{i=1}^4 U_i$

TABLE I: The labels  $A$ ,  $B$ ,  $C$  and  $D$  specify the vertices of the digraph, the entries  $R_i$  are the number of vehicles in the sites, and the  $Q_i$  and  $P_i$  are the numbers of vehicles trafficking along the roads, going from one vertex to the other. The sum of the entries of each row and column are  $U_i$ , and  $T_i$  respectively.  $Y$  is the total number of vehicles in the network.

The core of the array of Table I is a  $4 \times 4$  matrix

$$\mathbb{A} = \begin{pmatrix} R_1 & P_1 & P_5 & Q_4 \\ Q_1 & R_2 & P_2 & P_6 \\ Q_5 & Q_2 & R_3 & P_3 \\ P_4 & Q_6 & Q_3 & R_4 \end{pmatrix}, \quad (1)$$

that represents the distribution of the vehicles observed and counted in an *ad hoc* interval of time (at sites and roads), that could also be an average over previous observations.

Assuming that we want to forecast the evolution of the distribution of vehicles in the network day by day, we adopt the following causal interpretation based on the hypothesis that the vehicles distribution evolves according to a discrete time Markov chain: (a) The entries in the main diagonal in matrix (1),  $A_{ii} = R_i$ , stand for the numbers of vehicles at site  $i$ ; (b) the off diagonal entries  $A_{ij}$  ( $i \neq j$ ) are for the number of vehicles that left site  $i$  and are in their way to site  $j$ . For a period of 24 hours the sum of the entries of row  $i$ ,  $U_i = \sum_j A_{ij}$  represents the number of vehicles that remained in or left the site  $i$  driving to the other sites. Complementarily,  $T_j = \sum_i A_{ij}$  represents the number of vehicles that are in site  $j$  plus those that are in their way to arrive at it, having departed from the other sites. So, in some way, we may consider that matrix (1) represents a continuous observation for a 24-hour period, or an average over previous observations.

To forecast the day by day traffic flux of vehicles in the network for  $n$  days we assume a discrete time evolution, and construct, from matrix (1), the  $n$ -step Markov matrix  $\mathbb{M}^n$  – with the exponent assuming positive integers – that should permit to make probabilistic predictions about the future flux and distributions of the vehicles. This procedure is considered being linear since the entries in  $\mathbb{M}$  are not supposed to be  $n$ -dependent.

### III. THE STOCHASTIC MATRIX

Operationally, we map matrix (1) into a stochastic matrix (SM) that will be used as the generator of the evolution for the distribution of vehicles at subsequent moments of observation. We normalize the entries in each row of matrix (1), defining the parameters for the first row as

$$r_1 = \frac{R_1}{U_1}, \quad p_1 = \frac{P_1}{U_1}, \quad p_5 = \frac{P_5}{U_1}, \quad q_4 = \frac{Q_4}{U_1}; \quad (2)$$

and the same goes for the other rows, which are used to construct the transition probability matrix or SM

$$\mathbb{M} = \begin{pmatrix} r_1 & p_1 & p_5 & q_4 \\ q_1 & r_2 & p_2 & p_6 \\ q_5 & q_2 & r_3 & p_3 \\ p_4 & q_6 & q_3 & r_4 \end{pmatrix}, \quad (3)$$

because the sum of the (non-negative) entries of each row is 1. The sum of the entries of each column is not necessarily 1, however, if additionally the sum of the entries of each column happens also to be 1 then the matrix is said to be *doubly stochastic*. In more realistic instances the entries of a Markov matrix, as (3), can be constructed with empirical data collected from previous observations and

$$\mathbb{U}^T(0) = (U_1(0) \ U_2(0) \ U_3(0) \ U_4(0)), \quad (4)$$

represents an initial state for the distribution of vehicles; the superscript T stands for transposition. The  $n$ -step Markov chain rules the evolution of state (4)

$$\mathbb{U}^T(0)\mathbb{M}^n = \mathbb{U}^T(n), \quad (5)$$

that forecasts the distribution of vehicles at the  $n$ -th day, or any other chosen interval of time. For instance, we choose the vector  $\mathbb{U}^T(1)$  to forecast the number of vehicles,  $U_j(1)$ , 24 hours later.

### A. Properties of stochastic matrices

(1) The sum of the components of the vector  $\mathbb{U}(n)$  is a conserved quantity,

$$\sum_i U_i(n) = \sum_i U_i(0) \quad n = 1, 2, 3, \dots \quad (6)$$

because  $\mathbb{M}^n$  is still a stochastic matrix [34].

(2) The non-negative stochastic square matrix  $\mathbb{M} \geq 0$ , of dimension  $N$ , has eigenvalues  $\lambda_1 = 1$  and  $|\lambda_2|, |\lambda_3|, \dots, |\lambda_N| < 1$ , therefore  $\lim_{n \rightarrow \infty} |\lambda_k|^n = 0$ , for  $k \neq 1$ .  $\lambda_1 = 1$  is known in the literature as the Perron-Frobenius (PF) eigenvalue [32].

(3) Assuming that all the eigenvalues have multiplicity 1, with the linearly independent eigenvectors of matrix  $\mathbb{M}$ ,  $\mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \dots, \mathbb{X}^{(N)}$ , we construct the matrix

$$\mathbb{Q} = (\mathbb{X}^{(1)} \mathbb{X}^{(2)} \dots \mathbb{X}^{(N)}), \quad (7)$$

and write the decomposition of  $\mathbb{M}$  and its powers as

$$\mathbb{M} = \mathbb{Q}\mathbb{D}\mathbb{Q}^{-1}, \quad \text{and} \quad \mathbb{M}^n = \mathbb{Q}\mathbb{D}^n\mathbb{Q}^{-1}, \quad (8)$$

where  $\mathbb{D}$  is a diagonal matrix whose entries are the eigenvalues and  $\text{Tr}\mathbb{M}^n = \sum_{k=1}^N (\lambda_k)^n$  since the eigenvalue equation  $\mathbb{M}^n \mathbb{X}^{(k)} = (\lambda_k)^n \mathbb{X}^{(k)}$  holds. The spectral decomposition of  $\mathbb{M}^n$  in terms of the stationary state and the decaying modes is

$$\mathbb{M}^n = \mathbb{C}_1 + \sum_{l=2}^N (\lambda_l)^n \mathbb{C}_l. \quad (9)$$

The matrix  $\mathbb{C}_1$  (associated with the PF eigenvalue) is a SM, being the stationary state of the evolution; the other matrices  $\mathbb{C}_l$ ,  $l > 1$ , are not stochastic, having however the property  $\text{Tr}(\mathbb{C}_l) = 1$ . Since  $|\lambda_l| < 1$ , then  $\lim_{n \rightarrow \infty} \mathbb{M}^n = \mathbb{C}_1$ , which is the asymptotic irreversible stationary matrix,  $\det(\mathbb{C}_1) = 0$ . The eigenvalues  $\lambda_l$  can be associated with characteristic relaxation times: we write  $\lambda_l = \text{sgn}(\lambda_l) |\lambda_l|$ , where  $\text{sgn}(\cdot)$  is the sign of the argument. Therefore, we define the characteristic decay time as  $T_l = -(\ln |\lambda_l|)^{-1}$ , such that Eq. (9) can be written as

$$\mathbb{M}^n = \mathbb{C}_1 + \sum_{l=2}^N [\text{sgn}(\lambda_l)]^n \mathbb{C}_l \exp\left(-\frac{n}{T_l}\right). \quad (10)$$

Thereafter, one can write the evolution of the vector  $\mathbb{U}^T(0)$ ,  $\mathbb{U}^T(n) = \mathbb{U}^T(0)\mathbb{M}^n$  as

$$\mathbb{U}^T(n) = \mathbb{U}^T(0)\mathbb{C}_1 + \sum_{l=2}^N [\text{sgn}(\lambda_l)]^n (\mathbb{U}^T(0)\mathbb{C}_l) \exp\left(-\frac{n}{T_l}\right), \quad (11)$$

where  $\mathbb{U}^T(0)\mathbb{C}_1 = \lim_{n \rightarrow \infty} \mathbb{U}^T(n)$  is the asymptotic distribution of vehicles and  $[\text{sgn}(\lambda_l)]^n \times (\mathbb{U}^T(0)\mathbb{C}_l) \exp(-n/T_l)$  are  $(N-1)$  partial distributions in the transient regimes and the matrices can have negative entries, although the entries of the vectors  $\mathbb{U}^T(0)\mathbb{C}_1$  and  $\mathbb{U}^T(n)$  are positive. We illustrate the theory through an example.

**Example 1:** Let consider the stochastic matrix

$$\mathbb{M} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 4/10 & 3/10 & 3/10 \end{pmatrix}, \quad (12)$$

whose eigenvectors and eigenvalues are

$$\mathbb{X}^{(1)} = \begin{pmatrix} 0.577350 \\ 0.577350 \\ 0.577350 \end{pmatrix}, \quad \lambda_1 = 1, \quad \text{stationary mode}, \quad (13a)$$

$$\mathbb{X}^{(2)} = \begin{pmatrix} 0.576331 \\ -0.802915 \\ -0.152215 \end{pmatrix}, \quad \lambda_2 = 0.367945, \quad \text{decaying mode}, \quad (13b)$$

$$\mathbb{X}^{(3)} = \begin{pmatrix} 0.660268 \\ 0.039495 \\ -0.749991 \end{pmatrix}, \quad \lambda_3 = -0.067945, \quad \text{decaying mode}. \quad (13c)$$

The matrix  $\mathbb{M}$  is regular because  $\mathbb{M}^2$  is already positive (all entries are positive). We construct the matrix  $(\mathbb{X}^{(1)} \mathbb{X}^{(2)} \mathbb{X}^{(3)})$ ,

$$\mathbb{Q} = \begin{pmatrix} 0.577350 & 0.576331 & 0.660268 \\ 0.577350 & -0.802915 & 0.039495 \\ 0.577350 & -0.152215 & -0.749991 \end{pmatrix}, \quad (14)$$

and  $\mathbb{D} = \text{Diag}[1.0, 0.367945, -0.067945]$ . The matrix  $\mathbb{M}^n = \mathbb{Q}\mathbb{D}^n\mathbb{Q}^{-1} = (\mathbb{Y}^{(1,n)} \mathbb{Y}^{(2,n)} \mathbb{Y}^{(3,n)})$  so that the three columns are

$$\mathbb{Y}^{(1,n)} = \underbrace{\frac{11}{27} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{stationary mode}} + \underbrace{\begin{pmatrix} 0.304793 \\ -0.424622 \\ -0.326908 \end{pmatrix} (\lambda_2)^n + \begin{pmatrix} 0.287800 \\ 0.017215 \\ -0.326908 \end{pmatrix} (\lambda_3)^n}_{\text{decaying modes}} \quad (15)$$

$$\mathbb{Y}^{(2,n)} = \underbrace{\frac{2}{9} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{stationary mode}} + \underbrace{\begin{pmatrix} -0.544452 \\ 0.758503 \\ 0.143795 \end{pmatrix} (\lambda_2)^n + \begin{pmatrix} 0.322230 \\ 0.019275 \\ -0.366017 \end{pmatrix} (\lambda_3)^n}_{\text{decaying modes}} \quad (16)$$

$$\mathbb{Y}^{(3,n)} = \underbrace{\frac{10}{27} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{stationary mode}} + \underbrace{\begin{pmatrix} 0.239659 \\ -0.333880 \\ -0.063296 \end{pmatrix} (\lambda_2)^n + \begin{pmatrix} -0.610029 \\ -0.036490 \\ 0.692926 \end{pmatrix} (\lambda_3)^n}_{\text{decaying modes}}. \quad (17)$$

We further write the spectral decomposition of matrix (12) in terms of the modes or regimes  $\mathbb{M}^n = \mathbb{C}_1 + \mathbb{C}_2 (\lambda_2)^n + \mathbb{C}_3 (\lambda_3)^n$ , where the three matrices and their determinants are

$$\mathbb{C}_1 = \begin{pmatrix} 11/27 & 2/9 & 10/27 \\ 11/27 & 2/9 & 10/27 \\ 11/27 & 2/9 & 10/27 \end{pmatrix}; \quad \det \mathbb{C}_1 = 0, \quad (18)$$

$$\mathbb{C}_2 = \begin{pmatrix} 0.304793 & -0.544452 & 0.239659 \\ -0.424622 & 0.758503 & -0.333880 \\ -0.326908 & 0.143795 & -0.063296 \end{pmatrix}; \quad \det \mathbb{C}_2 = 1.07611 \times 10^{-7}, \quad (19)$$

$$\mathbb{C}_3 = \begin{pmatrix} 0.287800 & 0.322230 & -0.610029 \\ 0.017215 & 0.019275 & -0.036490 \\ -0.326908 & -0.366017 & 0.692926 \end{pmatrix}; \quad \det \mathbb{C}_3 = 3.24595 \times 10^{-13}. \quad (20)$$

While matrix  $\mathbb{C}_1$  does not have an inverse,  $\mathbb{C}_2$  and  $\mathbb{C}_3$  do have, but they are not stochastic. The decay characteristic times associated with the modes 2 and 3 are calculated as  $T_k = -(\ln |\lambda_k|)^{-1}$ , resulting in the values  $T_2 = 1.000178$  and  $T_3 = 0.371878$ , so mode 3 decays faster than mode 2, while  $T_1 = \infty$  for mode 1. Thus we can write

$$\mathbb{M}^n = \mathbb{C}_1 + \mathbb{C}_2 \exp\left(-\frac{n}{T_2}\right) + \mathbb{C}_3 \exp\left(-\frac{n}{T_3}\right). \quad (21)$$

An initial state vector,

$$\mathbb{U}(0) = \begin{pmatrix} 100 & 160 & 300 \end{pmatrix}, \quad (22)$$

will evolve as

$$\mathbb{U}^T(n) = \mathbb{U}^T(0) \mathbb{C}_1 + (\mathbb{U}^T(0) \mathbb{C}_2) \exp\left(-\frac{n}{T_2}\right) + (\mathbb{U}^T(0) \mathbb{C}_3) \exp\left(-\frac{n}{T_3}\right). \quad (23)$$

Thereafter, one can follow the evolution of the three components of the vector (23):

$$\mathbb{U}^T(0) \mathbb{C}_1 = \begin{pmatrix} 228 & 125 & 207 \end{pmatrix}, \quad (24a)$$

$$\mathbb{U}^T(0) \mathbb{C}_2 = \begin{pmatrix} -136 & 110 & -48 \end{pmatrix}, \quad (24b)$$

$$\mathbb{U}^T(0) \mathbb{C}_3 = \begin{pmatrix} -67 & -74 & 141 \end{pmatrix}, \quad (24c)$$

where  $\lim_{n \rightarrow \infty} \mathbb{U}^T(n) = \mathbb{U}^T(0) \mathbb{C}_1$  is the asymptotic component and  $\mathbb{U}^T(0) \mathbb{C}_l \times \exp(-n/T_l)$ ,  $l = 2, 3$ , are the components of the transient regimes that present negative numbers in some entries, although all the entries in the vectors  $\mathbb{U}^T(n)$  and  $\mathbb{U}^T(0) \mathbb{C}_1$ , Eqs. (23) and (24a), are positive. The sum of the entries of  $\mathbb{U}^T(0) \mathbb{C}_1$  is 560 which corresponds to the conserved sum of the entries of the vector (22), whereas the sums of the components of the vectors (24b) and (24c),  $\sum_{i=1}^3 (\mathbb{U}^T(0) \mathbb{C}_2)_i = -74$  and  $\sum_{i=0}^3 (\mathbb{U}^T(0) \mathbb{C}_3)_i = 0$  respectively, do not need to be necessarily positive, that we interpret as virtual distributions. As  $n$  increases the entries  $(\mathbb{U}^T(0) \mathbb{C}_l) \exp(-n/T_l)$  go to zero due to the exponential decay factor and only the asymptotic vector, Eq. (24a), survives.

(4) About the trace operation

$$\lim_{n \rightarrow \infty} \text{Tr} [\mathbb{M}^n] = 1 + \lim_{n \rightarrow \infty} \sum_{k=2}^N (\lambda_k)^n = 1. \quad (25)$$

The same holds true for doubly stochastic matrices.

(5) At the limit  $n \rightarrow \infty$  the stationary matrix is

$$\mathbb{M}_\infty = \lim_{n \rightarrow \infty} \mathbb{M}^n = \mathbb{C}_1 = \begin{pmatrix} m_1 & m_2 & \cdots & m_{N-1} & m_N \\ m_1 & m_2 & \cdots & m_{N-1} & m_N \\ \vdots & \vdots & & \vdots & \vdots \\ m_1 & m_2 & \cdots & m_{N-1} & m_N \\ m_1 & m_2 & \cdots & m_{N-1} & m_N \end{pmatrix} \quad (26)$$

with  $\sum_{j=1}^N m_j = 1$ , the set of numbers  $\{m_j\}$  are the asymptotic values (stationary), and they depend on the initial entries of matrix  $\mathbb{M}$ . At infinity ( $n \rightarrow \infty$ ) the initial matrix  $\mathbb{M}$  is not retrievable from matrix  $\mathbb{C}_1$ , therefore it is not invertible and its eigenvalues are  $\{1, 0, 0, \dots, 0\}_N$ ; the entries of each column are the same. As matrix (26) is the asymptotic matrix, an evolved non-negative vector,  $\mathbb{X}_0^T = (x_1 \ x_2 \ \cdots \ x_{N-1} \ x_N)$ ,  $\mathbb{X}_\infty = \mathbb{X}_0^T \mathbb{C}_1$  loses all the memory about the entries, keeping the sum  $\sum_{i=1}^N x_i = Y$  as a single reminder,

$$\mathbb{X}_\infty^T = \mathbb{X}_0^T \mathbb{C}_1 = Y \mathbb{Z}^T, \quad (27)$$

where

$$\mathbb{Z}^T = \begin{pmatrix} m_1 & m_2 & \cdots & m_{N-1} & m_N \end{pmatrix} \quad (28)$$

is a row of matrix (26). As so, the knowledge of the matrix  $\mathbb{M}$  allows to determine the asymptotic matrix  $\mathbb{C}_1$ . However, the inverse is not possible, the knowledge of matrix  $\mathbb{C}_1$  does not permit the determination of matrix  $\mathbb{M}$ . That is the essence of the irreversible evolution. The vector (28) is an eigenstate of the Markov chain matrix  $\mathbb{M}$ , Eq. (3), with eigenvalue 1:  $\mathbb{Z}^T \mathbb{M} = \mathbb{Z}^T$ . This can be verified in the above example, Eqs. (12) and (18),

$$\begin{pmatrix} \frac{11}{27} & \frac{2}{9} & \frac{10}{27} \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 4/10 & 3/10 & 3/10 \end{pmatrix} = \begin{pmatrix} \frac{11}{27} & \frac{2}{9} & \frac{10}{27} \end{pmatrix}. \quad (29)$$

(6) In the numerical examples to be worked out below the eigenvalues of the matrices have multiplicity 1, nevertheless the math and the physical analysis can be extended to the situation where eigenvalues have multiplicity higher than 1. As an illustration about this point we consider here an interesting behavior that affects the time decay in the case of occurrence of a degenerate eigenvalue (in comparison to the situation of no degeneracy). Pointedly, within a closed network, it causes a delay in the time the flux of vehicles takes to evolve to the stationary matrix  $\lim_{n \rightarrow \infty} \mathbb{M}^n$ . In this case the diagonalization of the matrix  $\mathbb{M}$  – the decomposition (8) – that produces the diagonal matrix  $\mathbb{D}$ , is not anymore possible because there are fewer linearly independent eigenvectors than the dimension  $N$  of the matrix  $\mathbb{M}$ . Notwithstanding, we can write the decomposition in a form similar to (8),  $\mathbb{M} = \mathbb{R}\mathbb{F}\mathbb{R}^{-1}$ , which is the so-called *Jordan canonical form*, where the matrix  $\mathbb{R}$  is different from  $\mathbb{Q}$ , and the matrix  $\mathbb{F}$ , besides having the eigenvalues of  $\mathbb{M}$  in the diagonal line, will contain, additionally, in the adjacent line parallel to the main diagonal, the number 1 in the entries of the blocks containing the degenerate eigenvalues, while the other entries are filled with zeros. The theory is due to the french mathematician Camille Jordan. For a rigorous mathematical presentation of the matter we recommend the reader to consult, for instance, Ref. [35].

The decomposition of a stochastic matrix in the Jordan form can be cast as

$$\mathbb{M} = \mathbb{R}\mathbb{D}\mathbb{R}^{-1} + \mathbb{L}, \quad \text{and} \quad \mathbb{M}^n = \mathbb{R}\mathbb{D}^n\mathbb{R}^{-1} + f(n)\mathbb{L}, \quad (30)$$

where  $\mathbb{D}$  is still a diagonal matrix containing the eigenvalues,  $\mathbb{L} = \mathbb{R}\mathbb{O}\mathbb{R}^{-1}$  with  $\mathbb{O}$  the matrix containing 1's (and 0's) in a secondary diagonal of matrix  $\mathbb{F}$  and  $f(n)$  is a function of the discrete time. We illustrate this case through an example.

**Example 2:** Let us consider the doubly stochastic matrix

$$\mathbb{M} = \begin{pmatrix} 2/5 & 1/2 & 1/10 \\ 3/10 & 3/10 & 2/5 \\ 3/10 & 1/5 & 1/2 \end{pmatrix}; \quad \det \mathbb{M} = 1/100. \quad (31)$$

$$(32)$$

whose eigenvectors and eigenvalues are

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \leftrightarrow \chi_{PF} = 1, \quad \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \leftrightarrow \chi_2 = \frac{1}{10}, \text{ multiplicity } 2, \quad (33)$$

$\chi_{PF}$  is the Perron-Frobenius eigenvalue and  $\chi_2$  is degenerate, with only one linearly independent eigenvector. The decomposition of matrix (56) in the Jordan form is

$$\mathbb{M} = \mathbb{R}\mathbb{F}\mathbb{R}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 5/2 & -1/2 \\ 1 & -5/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 1 & \chi_2 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/5 & -1/5 \\ 2/3 & -1/3 & -1/3 \end{pmatrix} \quad (34)$$

where the middle matrix is not diagonal although the eigenvalues are in the main diagonal. We can rewrite the matrix (34) as

$$\mathbb{M} = \mathbb{R}\mathbb{D}\mathbb{R}^{-1} + f(1)\mathbb{L} = \mathbb{C}_1 + \chi_2\mathbb{C}_2 + \chi_2\mathbb{C}_3 \quad (35)$$

where the matrices  $\mathbb{C}_i$  are

$$\mathbb{C}_1 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbb{C}_2 = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}, \quad \mathbb{C}_3 = \begin{pmatrix} 0 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}. \quad (36)$$

For an  $n$ -step process we have  $\mathbb{M}^n = \mathbb{C}_1 + (\chi_2)^n \mathbb{C}_2 + n(\chi_2)^n \mathbb{C}_3$ . Setting  $\chi_2 = \exp(-1/T_2)$  we get

$$\mathbb{M}^n = \mathbb{C}_1 + \exp\left(-\frac{n}{T_2}\right) \mathbb{C}_2 + n \exp\left(-\frac{n}{T_2}\right) \mathbb{C}_3 \quad (37)$$

with the characteristic decay time  $T_2 = -(\ln \chi_2)^{-1} = (\ln 10)^{-1} \approx 0.43$  for the exponential decay modes. Due to the factor  $n$  that multiplies the exponential in the third term, it is going to decay at a lower pace than the second, thus dominating the evolution of  $\mathbb{M}^n$  toward the stationary matrix  $\mathbb{C}_1$  as  $n \rightarrow \infty$ .

### B. Another generator of the evolution

We can define another generator of the evolution, that shows a close similarity to the hamiltonian of classical or quantum mechanics. From Eq. (5) we can write  $(\mathbb{U}(n+1))^T - (\mathbb{U}(n))^T = -(\mathbb{U}(n))^T (\mathbf{1} - \mathbb{M})$ . For  $n \gg 1$ , the “differential equation” is

$$\frac{d\mathbb{U}(n)}{dn} = -\mathbb{H}\mathbb{U}(n), \quad (38)$$

where the generator of the evolution,  $\mathbb{H} = \mathbf{1} - \mathbb{M}^T$ , is sometimes called “hamiltonian”, which is a non-negative matrix. Thus, in this form the evolution is dictated by the properties of the matrix  $\mathbb{H}$ .

### C. Entropy

The complexity of the flow of vehicles in a network, as the one represented by the digraph (1), has a measure that can be evaluated by calculating the entropy at each moment  $n$ . Since  $\sum_{j=1}^N (\mathbb{M}^n)_{ij} = 1$  ( $N$  stands for the dimension of the square matrix), for the SM we define the partial entropy associated to each row  $i$  as

$$S_i(n) = - \sum_{j=1}^N (\mathbb{M}^n)_{i,j} \ln (\mathbb{M}^n)_{i,j} \quad \text{for } i = 1, \dots, N; \quad (39)$$

noting that if no vehicle migrates from one city to the other, then  $(\mathbb{M}^n)_{ij} = \delta_{ij}$  and  $S_i(n) = 0$ . We can also define a measure of the global complexity as an arithmetic mean over all the sites,

$$G(n) = \frac{1}{N} \sum_{i=1}^N S_i(n) = -\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N (\mathbb{M}^n)_{ij} \ln (\mathbb{M}^n)_{ij}, \quad (40)$$

outlining a *global mean entropy*.

## IV. MODEL I: ILLUSTRATION WITH NUMBERS

In the four sites network (digraph of Fig. 1) we assume that there are, initially, 10 000, 2 500, 11 000 and 13 000 vehicles; the matrix (1) with numbers in its entries is diagonal and is more conveniently expressed as the vector (4),

$$\mathbb{U}^T(0) = (10000 \ 2500 \ 11000 \ 13000), \quad (41)$$

and the total number of vehicles in the network is

$$Y = \sum_{i=1}^4 U_i(0) = 36\ 500. \quad (42)$$

At a given moment the vehicles begin to circulate through the highways and after few days of observation the average number of the vehicles that remain circulating (or parked) within each city (the vertices) are assumed *ad hoc* to be 3 000, 1 000, 4 000, and 4 500, while the remaining vehicles are on their way traveling to the other cities, characterized in the digraph by the values associated to the edges. We put the available information in the matrix

$$\mathbb{A}(1) = \begin{pmatrix} 3000 & 1500 & 2500 & 3000 \\ 500 & 1000 & 500 & 500 \\ 3000 & 1500 & 4000 & 2500 \\ 4000 & 1500 & 3000 & 4500 \end{pmatrix}, \quad (43)$$

where in the diagonal entries one finds the number of vehicles in each city and the off-diagonal entries stand for the number of those traveling from one city (line  $i$ ) to another (column  $j$ ), and  $\text{Tr}(\mathbb{A}(1)) = 12\ 500$  is the number of vehicles still in the sites. The number of vehicles in each row ( $U_k(0)$ ) are given by the vector (41).



### A. The Stochastic Matrix

We assume that the matrix (43) is the seed that permits to construct another matrix, in a suited form, to be used for forecasting the circulation of the vehicles in future moments. This hypothesis is framed formally by normalizing each row. The result is the SM

$$\mathbb{M} = \begin{pmatrix} 3/10 & 3/20 & 1/4 & 3/10 \\ 1/5 & 2/5 & 1/5 & 1/5 \\ 3/11 & 3/22 & 4/11 & 5/22 \\ 4/13 & 3/26 & 3/13 & 9/26 \end{pmatrix}, \quad (44)$$

whose eigenvalues are approximately

$k$	1	2	3	4
$\lambda_k$	1.0	0.27	0.13	0.01

(45)

The first eigenvalue,  $\lambda_1 = 1.0$ , is the PF one while the others are quite smaller than 1, this is an indication that under this Markov evolution the traffic should stabilize after very few steps. As  $\det \mathbb{M} = (2600)^{-1} < 1$ , then  $\lim_{n \rightarrow \infty} \det (\mathbb{M}^n) = 0$  and  $\lim_{n \rightarrow \infty} \text{Tr} (\mathbb{M}^n) = 1$ .

Now we calculate the number of vehicles in each city plus those that left (in a period of 24 hours). The prediction of the distribution for the following days is given by Eq. (5), and the total number of vehicles is a conserved quantity,  $\lim_{n \rightarrow \infty} \sum_i U_i(n) = 36\,500 = Y$ . The distribution changes with  $n$  until the stabilization of the flow is attained, in less than 4 steps (days) later. This short period of time could be guessed due to the wide difference between the PF eigenvalue ( $\lambda_1=1$ ) and  $\lambda_2$ , see the frame (45). The network presents three characteristic decay times to attain the stationary distribution

$i$	1	2	3	4
$T_i$	$\infty$	0.76	0.49	0.22

(46)

where the second decay time,  $T_2$ , dictates the trend of the decay. In Fig. 2 we plot the sequences of the distributions  $U_i(n)$  that swiftly stabilize to asymptotic ones. For  $\mathbb{U}(0) = (10000 \ 2500 \ 11000 \ 13000)$  we get  $\mathbb{U}(\infty) \approx \mathbb{U}(6) = (10097 \ 6659 \ 9702 \ 10042)$ .

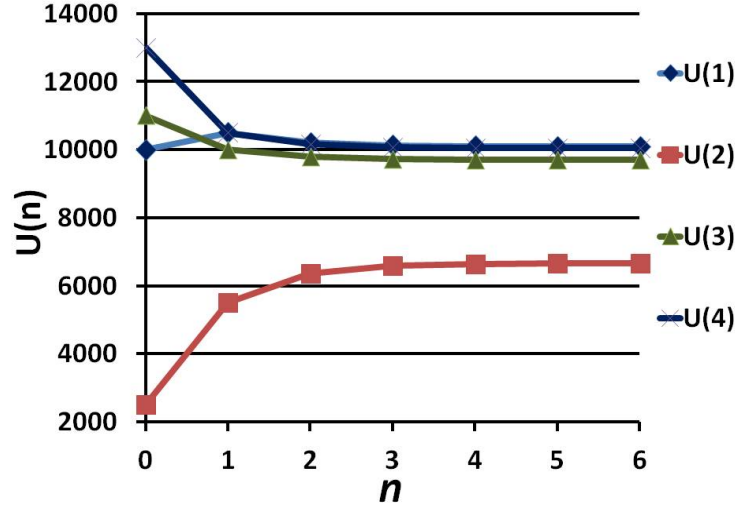


FIG. 2: The sequences of the distributions  $U_i(n)$ .

### B. Entropy

The global mean entropy (40) takes numerical values given in Table II,

$n$	1	2	3	4	5	6
$G(n)$	1.333	1.372	1.373	1.373	1.373	1.373

TABLE II: Numerical values for the global mean entropy, Eq. (40).

and in Fig. 3 the diamond shaped marks represent the calculated values of  $G(n)$  of Table II; the solid line only links the points.

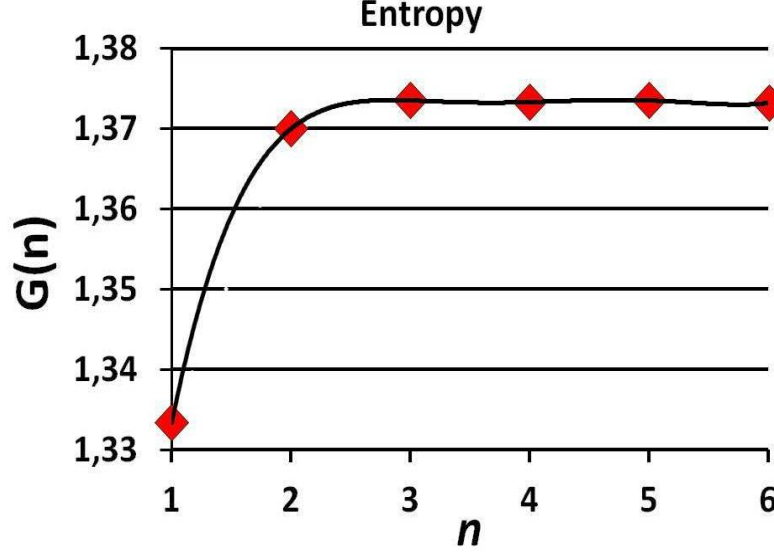


FIG. 3: The global mean entropy as defined in Eq. (40).

The stabilization of the distribution of vehicles in each city, and the stationarity of the flow is reflected in the behavior of the global mean entropy which increases swiftly and then stabilizes at a maximum value, that we could call as the traffic “thermalization” of vehicles.

## V. MODEL II: INTERCITY TRAFFIC WITH INPUT (SOURCE) AND OUTPUT (SINK) OF VEHICLES

Here we introduce two physical modifications in the intercity traffic model studied in the preceding sections: (a) there is a daily input of new vehicles from site  $E$  to city  $A$ , that are subsequently absorbed into the whole network and (b) there is also an output of old vehicles that are taken off the circuit from the same city  $A$  and are stocked in the site  $F$ . These new features are drawn in the digraph with additional edges and loops, labeled as  $v_1$  and  $v_2$ ,  $r_5$  and  $r_6$ , respectively, as shown in Fig. 4. In this digraph the labels  $A, B, C, D, E$  and  $F$  classify the vertices, whereas the  $r_i$  are values for the loops and  $q_i, p_i, s_1$  and  $v_1$  values for the edges. One expresses the network by the array in Table III. whose core is a  $6 \times 6$  matrix

$$\mathbb{B} = \begin{pmatrix} R_1 & P_1 & P_5 & Q_4 & 0 & S_1 \\ Q_1 & R_2 & P_2 & P_6 & 0 & 0 \\ Q_5 & Q_2 & R_3 & P_3 & 0 & 0 \\ P_4 & Q_6 & Q_3 & R_4 & 0 & 0 \\ V_1 & 0 & 0 & 0 & R_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_6 \end{pmatrix}, \quad (47)$$

that represents the distribution of the vehicles as observed and counted in an *ad hoc* interval of time, 24 hours for example, or being an average over previous observations. The matrix (1) is a submatrix of matrix (47).

As the time goes on the distribution of vehicles, in matrix (47), changes due to their circulation. The sum of the entries of row  $i$ ,  $W_i(0) = \sum_j B_{ij}$ , gives the number of vehicles that are in city  $i$  plus those that left it having as destination all the other cities, excluding site  $F$ , which is a depot of the vehicles removed from circulation. The total

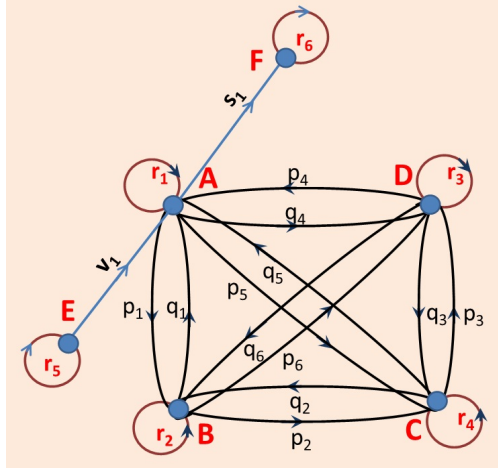


FIG. 4: Two physical modifications in the intercity traffic as drawn in Fig. 1

Vertices	A	B	C	D	E	F	Sum of the line entries
A	$R_1$	$P_1$	$P_5$	$Q_4$	0	$S_1$	$W_1$
B	$Q_1$	$R_2$	$P_2$	$P_6$	0	0	$W_2$
C	$Q_5$	$Q_2$	$R_3$	$P_3$	0	0	$W_3$
D	$P_4$	$Q_6$	$Q_3$	$R_4$	0	0	$W_4$
E	$V_1$	0	0	0	$R_5$	0	$W_5$
F	0	0	0	0	0	$R_6$	$W_6$
Sum of the column entries	$Z_1$	$Z_2$	$Z_3$	$Z_4$	$Z_5$	$Z_6$	$\bar{Y}$

TABLE III: The labels  $A, B, C, D, E$  and  $F$  specify the vertices in the digraph, the entries  $Q_i, P_i, V_1$  and  $S_1$  stand for the numbers of vehicles trafficking along the roads, from one site to the other, and the  $R_i$ 's are the numbers of vehicles present at each site. The sum of the entries of each row and column are  $W_i$  and  $Z_i$  respectively.  $\bar{Y} = \sum_{i=1}^6 W_i = \sum_{i=1}^6 Z_i$  is the total number of vehicles in the network.

number of vehicles in the network,  $\bar{Y}$ , is conserved. The numbers  $W_i(0)$  can be cast as a vector,

$$\mathbb{W}^T(0) = (W_1(0) \ W_2(0) \ W_3(0) \ W_4(0) \ W_5(0) \ W_6(0)). \quad (48)$$

#### A. The stochastic matrix

The dynamical evolution is ruled by a Markov matrix associated to matrix (47) which is constructed by normalizing the entries in each row, resulting in

$$\mathbb{N} = \begin{pmatrix} r_1 & p_1 & p_5 & q_4 & 0 & s_1 \\ q_1 & r_2 & p_2 & p_6 & 0 & 0 \\ q_5 & q_2 & r_3 & p_3 & 0 & 0 \\ p_4 & q_6 & q_3 & r_4 & 0 & 0 \\ v_1 & 0 & 0 & 0 & r_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (49)$$

where the first row is

$$r_1 = \frac{R_1}{W_1}, \quad p_1 = \frac{P_1}{W_1}, \quad p_5 = \frac{P_5}{W_1}, \quad q_4 = \frac{Q_4}{W_1}, \quad s_1 = \frac{S_1}{W_1}; \quad (50)$$

the sum of the entries of each row is 1 and the same goes for the other rows. The evolution of an initial state is calculated as

$$\mathbb{W}^T(n) = \mathbb{W}^T(0) \mathbb{N}^n. \quad (51)$$

## VI. MODEL II: ILLUSTRATING THE MODEL WITH NUMBERS

To illustrate the model we consider the matrix (49) with arbitrarily chosen entries,

$$\mathbb{B}(1) = \begin{pmatrix} 3000 & 1500 & 2500 & 3000 & 0 & 50 \\ 500 & 1000 & 500 & 500 & 0 & 0 \\ 3000 & 1500 & 4000 & 2500 & 0 & 0 \\ 4000 & 1500 & 3000 & 4500 & 0 & 0 \\ 100 & 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_6 \end{pmatrix}, \quad (52)$$

where we kept the same values of the entries of the  $4 \times 4$  matrix (43) but is now enlarged, with dimensions  $6 \times 6$ , and with new non-null entries,  $B_{16} = 50$ ,  $B_{51} = 100$ ,  $B_{55} = 1000$ ; the entry  $B_{66} = R_6$  is an arbitrary number, that we set as 0 since it is not important within the dynamics. Summing the entries in each row of matrix (52) we have the vector

$$\mathbb{W}^T(0) = (10050 \quad 2500 \quad 11000 \quad 13000 \quad 1100 \quad 0) \quad (53)$$

or as a diagonal matrix  $\mathbb{B}(0) = \text{Diag}[10050, 2500, 11000, 13000, 1100, 0]$  and the total number of vehicles is  $\bar{Y} = 37\,650$ . Comparing with Model I, the increment in the number of vehicles in vector (53) is small compared to those in the string (41), namely an increment of 1 150 to the previous 36 500.

### A. The Stochastic Matrix

From the matrix (52) we construct the stochastic matrix

$$\mathbb{N} = \begin{pmatrix} 300/1005 & 150/1005 & 250/1005 & 300/1005 & 0 & 5/1005 \\ 1/5 & 2/5 & 1/5 & 1/5 & 0 & 0 \\ 3/11 & 3/22 & 4/11 & 5/22 & 0 & 0 \\ 4/13 & 3/26 & 3/13 & 9/26 & 0 & 0 \\ 1/11 & 0 & 0 & 0 & 10/11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (54)$$

whose eigenvalues are, approximately,

$k$	1	2	3	4	5	6
$\lambda_k$	1.0	0.999	0.909	0.266	0.132	0.011

(55)

The first eigenvalue is the PF while the second and third ones are close to 1, making the traffic to take much more time to attain a stationary circulation than in the former model where the traffic goes stationary in 3 or 4 steps; these eigenvalues can be compared to those in frame (45).

The distribution of vehicles evolves as  $\mathbb{W}^T(n) = \mathbb{W}^T(0)\mathbb{N}^n$ , with  $\mathbb{W}^T(0)$  from Eq. (53). For  $n = 2^{12}$  we get  $\mathbb{W}^T(2^{12}) = (37 \quad 24 \quad 36 \quad 37 \quad 0 \quad 37516)$ . The last entry stands for the 37516 vehicles that went out of circulation (a fraction 1/201, per day, of all vehicles from site *A* go to site *F*) that accumulated continuously and increasingly with  $n$  whereas the number of vehicles still in circulation diminishes. The fifth entry stands for the supply of new vehicles into the network: there were initially 1 000 vehicles at site *E*, with a daily supply into the network *ABCD* of 1/11 fraction of vehicles in the car yard *E* (the supply lasted few days). We note that after  $2^{12}$  steps only a tiny fraction of vehicles (134), 0.36%, continues circulating within the network *ABCD*. In Fig. 5 we plot the evolution of the entries of  $\mathbb{W}(n)$ , considering in the abscissa  $k \equiv (\log_2 n) + 1$ , for  $n = 2^0, 2^1, \dots, 2^{12}$ , such that  $k = 1, 2, \dots, 13$ . Thus, after an initial increase of the number of vehicles circulating in the network *ABCD* there is a stabilization, see Fig. 6, then a slow decrease in their number occurs, however after a quite long time compared to the one that it takes to attain equilibration in the model without source and sink.

We note that for the digraph in Fig. 1 the asymptotic equilibrium of the evolution is attained nearly after four steps,  $n = 4$ , whereas for the network with source and sink, the digraph in Fig. 4, the tendency to the stationary state,  $\mathbb{W}(2^{12})$ , occurs after  $2^{12}$  steps, or for  $(\mathbb{N})^{2^{12}}$ . In short, in the former network model the stationarity is reached at a rate linear in time, while here it occurs at an exponential rate. The characteristic decay times  $\tilde{T}_k = -(\ln |\lambda_k|)^{-1}$  are

$k$	1	2	3	4	5	6
$\tilde{T}_k$	$\infty$	726.20	10.50	0.76	0.49	0.22

(56)

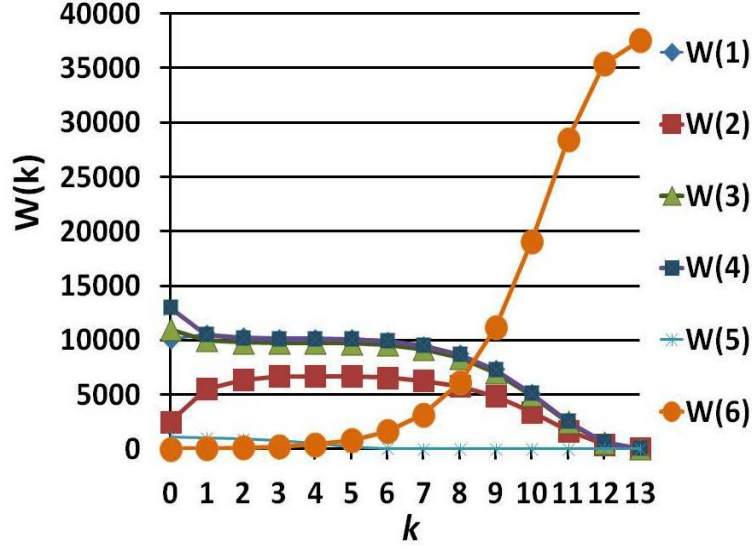


FIG. 5: The evolution of the number of vehicles in each entry of the vector  $\mathbb{W}(n)$ , at several moments, where we defined  $k = 1 + \log_2 n$  as the abscissa variable. We have set *ad hoc*  $k = 0$  for  $n = 0$ .

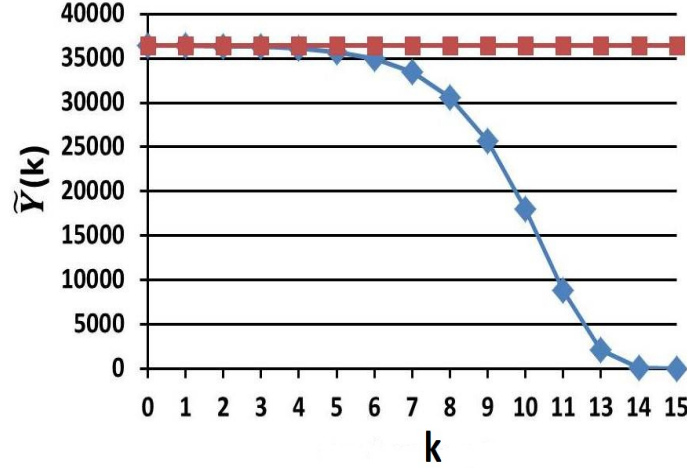


FIG. 6: The line in blue (diamonds marks) is the evolution of the number of vehicles within the network  $ABCD$ ,  $\tilde{Y}(n)$ , with  $k = \log_2 n + 1$  and for  $n = 0$  we have set  $k = 0$ . The red line (square marks) stands for the network without source and sink.

to be compared to the characteristic times in the frame (46). Since  $\tilde{T}_2/T_2 \approx 961$ ,  $\tilde{T}_3/T_3 \approx 13.9$ ,  $\tilde{T}_4/T_4 \approx 3.4$ , we perceive the huge ratio of the characteristic times  $\tilde{T}_2/T_2$  meaning that the introduction of a source and a sink in the  $ABCD$  network, for the specific chosen numbers, contribute to an average time (to attain the stationary state) that is about three orders of magnitude higher than the network without source and sink.

### B. Entropy

The global mean entropy, Eq. (40), for  $N = 6$ , at times  $n = 2^k$  with  $k = 0, 1, 2, \dots, 12$  is plotted in Fig. 7. Compared to Fig. 3 it shows a quite different trend, the entropy increases, attains a maximum value and then begins a monotonic decrease until zero, when all the vehicles accumulate at the site  $F$ , as they are being continuously collected.

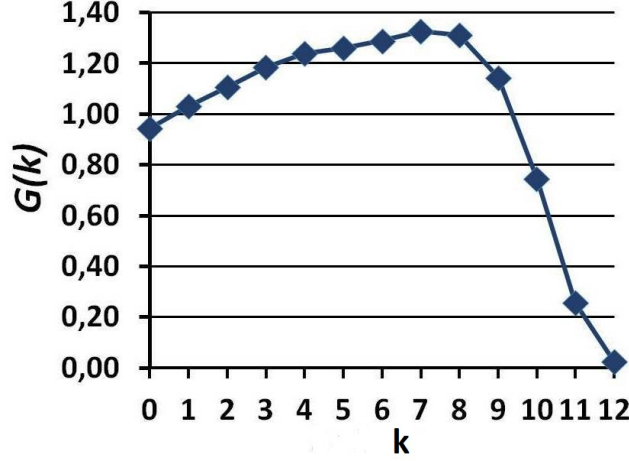


FIG. 7: Global mean entropy  $G(k)$  for the stochastic matrix  $\mathbb{N}^n$ , where  $k = \log_2 n$ .

## VII. NONLINEAR MODEL: MULTIPLE STATIONARY STATES AND CYCLIC CHANGES

More realistically one could not expect that the traffic flow should obey strictly a linear theory, even with sources and sinks in the network, since it leads to only one stationary state. Thus a nonlinear evolution approach seems more realistic because it introduces non-trivial changes in predicting the flow. In this model the entries of a stochastic matrix contain functions of  $n$ , as an intrinsic variable, and there is a freedom to choose functions and parameters in order to reproduce a flow displaying several stationary regimes in one day.

For the sake of illustration of the formal approach we consider a two-site network ( $A$  and  $B$ ) and two roads connecting them, with no sinks or sources. The stochastic matrix has dimension 2, for instance, and we can write it as

$$(\mathbb{M}(n))^n = \begin{pmatrix} g_1(n) & 1 - g_1(n) \\ g_2(n) & 1 - g_2(n) \end{pmatrix}^n, \quad (57)$$

whose entries are  $n$ -dependent and remain non-negative for any positive integer  $n$ ,  $0 \leq g_k(n) \leq 1$ . Imposing, additionally, that the functions  $g_k(n)$  change periodically with  $n$ , we choose arbitrarily  $g_1(n) = a_1 + b_1 \sin(\frac{2\pi n}{L})$  and set  $g_2(n) = a_2 + b_2 \cos(\frac{2\pi n}{L})$ , with  $a_1 = a_2 = 1/2$ ,  $b_1 = b_2 = -1/2$ , and, as it will be seen bellow, the integer  $L$  stands for the multiplicity of the stationary states. For the sake of illustration we adopt  $L = 3$ , which implies three asymptotic matrices of an  $n$ -step Markov evolution. This property of the nonlinear stochastic matrices enlarges the descriptive possibilities of physical systems in comparison to the linear models that allow only one stationary state. Such choices turn matrix (57) into

$$(\mathbb{M}(n))^n = \begin{pmatrix} \frac{1}{2} (1 - \sin \frac{2\pi n}{3}) & \frac{1}{2} (1 + \sin \frac{2\pi n}{3}) \\ \frac{1}{2} (1 - \cos \frac{2\pi n}{3}) & \frac{1}{2} (1 + \cos \frac{2\pi n}{3}) \end{pmatrix}^n, \quad (58)$$

and it is the dependence on  $n$  within the periodic functions that characterizes the nonlinearity. The matrix  $\mathbb{M}(n)$  is 3-cycle, meaning that for each element of the set  $n = \{3k, 3k+1, 3k+2\}$  (excluding  $n = 0$ ) the matrices repeat themselves as shown in the Table IV. This choice of the integers  $k$  maps one-to-one the sequence of the positive integers  $n$ .

$k$	$n = 3k$	$n = 3k + 1$	$n = 3k + 2$
0	0	1	2
1	3	4	5
2	6	7	8
$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE IV: The integers  $k$  map one-to-one the sequence of integers  $n$ .

The Markov matrices and decay modes eigenstates and eigenvalues are given in Table V, and their expansion in the several modes and their decay times are given in Table VI.

$n$ -step Markov matrices	Decay mode eigenstate and eigenvalue
$\mathbb{M}(3k+1) \doteq \mathbb{M}_1 = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{4} & \frac{1}{2} + \frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$	$\frac{1}{2\sqrt{4+\sqrt{3}}} \begin{pmatrix} -(\sqrt{3}+2) \\ 3 \end{pmatrix} \leftrightarrow -\frac{1}{4}(\sqrt{3}+1) = \lambda_1$
$\mathbb{M}(3k+2) \doteq \mathbb{M}_2 = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}}{4} & \frac{1}{2} - \frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$	$\frac{1}{2\sqrt{4-\sqrt{3}}} \begin{pmatrix} \sqrt{3}-2 \\ 3 \end{pmatrix} \leftrightarrow \frac{1}{4}(\sqrt{3}-1) = \lambda_2$
$\mathbb{M}(3k) \doteq \mathbb{M}_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}, \quad k \neq 0$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow \frac{1}{2} = \lambda_3$

TABLE V:  $n$ -step Markov matrices and decay modes eigenstates and eigenvalues. The PF eigenvalue is the same for the three cases,  $\lambda_{PF} = 1$ .

	$T_i = -(\ln \lambda_i )^{-1} \approx$
$\mathbb{M}_1^n = \mathbb{C}_{1,1} + \mathbb{C}_{2,1}e^{-\frac{n}{T_1}}$	2.62
$\mathbb{M}_2^n = \mathbb{C}_{1,2} + \mathbb{C}_{2,2}e^{i(\pi+i\frac{1}{T_2})n}$	0.59
$\mathbb{M}_3^n = \mathbb{C}_{1,3} + \mathbb{C}_{2,3}e^{-\frac{n}{T_3}}$	1.44

TABLE VI: Expansion in terms of the spectral modes (matrices) and decay characteristic times.

The PF eigenstates for the three matrices in Table V are the same,  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , and the  $\mathbb{C}_{l,i}$  matrices are given in Table VII.

$i$	$\mathbb{C}_{1,i}$	$\mathbb{C}_{2,i}$
1	$\frac{1}{\sqrt{3+5}} \begin{pmatrix} 3 & \sqrt{3}+2 \\ 3 & \sqrt{3}+2 \end{pmatrix}$	$\frac{1}{\sqrt{3+5}} \begin{pmatrix} \sqrt{3}+2 & -(\sqrt{3}+2) \\ -3 & 3 \end{pmatrix}$
2	$\frac{1}{5-\sqrt{3}} \begin{pmatrix} 3 & 2-\sqrt{3} \\ 3 & 2-\sqrt{3} \end{pmatrix}$	$\frac{1}{5-\sqrt{3}} \begin{pmatrix} 2-\sqrt{3} & -(2-\sqrt{3}) \\ -3 & 3 \end{pmatrix}$
3	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

TABLE VII: The  $\mathbb{C}_{1,i}$  matrices are stochastic, whereas the  $\mathbb{C}_{2,i}$  matrices are not and the sum of the entries in each row is null.

We now assume that initially the number of vehicles associated with the two cities,  $A$  and  $B$ , are  $a_0$  and  $b_0$ , represented by the statevector  $\begin{pmatrix} a_0 & b_0 \end{pmatrix}$ . For a daily prediction of the number of vehicles at three different moments of a day we set the rules in Table VIII and by moment we mean some, arbitrarily chosen, interval of time for counting

$i$	$r$ -th DAY evolution, $r = 1, 2, 3, \dots$	$n$
1	$\begin{pmatrix} a_r^{(1)} & b_r^{(1)} \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \end{pmatrix} \mathbb{M}_1 (\mathbb{M}_2 \mathbb{M}_3 \mathbb{M}_1)^{r-1}$	$3(r-1) + 1$
2	$\begin{pmatrix} a_r^{(2)} & b_r^{(2)} \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \end{pmatrix} \mathbb{M}_1 \mathbb{M}_2 (\mathbb{M}_3 \mathbb{M}_1 \mathbb{M}_2)^{r-1}$	$3(r-1) + 2$
3	$\begin{pmatrix} a_r^{(3)} & b_r^{(3)} \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \end{pmatrix} (\mathbb{M}_1 \mathbb{M}_2 \mathbb{M}_3)^r$	$3r$

TABLE VIII: Evolution of the number of vehicles,  $a_r^{(i)}$  and  $b_r^{(i)}$  at the  $r$ -th day, in the three stationary states  $i$  associated with sequential observations: 1 (morning), 2 (noon) and 3 (evening). The integer  $n$  stands for the sequence of moments for  $r$  days.

the number of vehicles passing by an established mark in a road, or the traffic flux, at three different times of a day, for instance, morning, noon and evening, to be compared with the predictions of the model.

Using properties of stochastic matrices the products of matrices in Table VIII can be written as

$$\mathbb{M}_1 (\mathbb{M}_2 \mathbb{M}_3 \mathbb{M}_1)^{r-1} = \mathbb{H}_{1,1} + \mathbb{H}_{2,1} (-1)^{r-1} e^{-\frac{r-1}{\tau_1}}, \quad (59a)$$

$$\mathbb{M}_1 \mathbb{M}_2 (\mathbb{M}_3 \mathbb{M}_1 \mathbb{M}_2)^{r-1} = \mathbb{H}_{1,2} + \mathbb{H}_{2,2} (-1)^{r-1} e^{-\frac{r-1}{\tau_1}}, \quad (59b)$$

$$(\mathbb{M}_1 \mathbb{M}_2 \mathbb{M}_3)^r = \mathbb{H}_{1,3} + \mathbb{H}_{2,3} (-1)^r e^{-\frac{r}{\tau_1}}, \quad (59c)$$

with  $r = 1, 2, \dots$ , and since the eigenvalues of the three matrices (59) coincide,  $\lambda_1 = -(1/16)$ , the decay time turns to be  $\tau_1 = -(\ln(1/16))^{-1} \approx 0.36$ . The  $\mathbb{H}$  matrices are

$$\mathbb{H}_{1,1} = \frac{1}{34} \begin{pmatrix} 21 - 3\sqrt{3} & 13 + 3\sqrt{3} \\ 21 - 3\sqrt{3} & 13 + 3\sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 0.46 & 0.54 \\ 0.46 & 0.54 \end{pmatrix} \quad (60a)$$

$$\mathbb{H}_{2,1} = \frac{1}{17} \begin{pmatrix} -(\frac{11}{4}\sqrt{3} + 2) & \frac{11}{4}\sqrt{3} + 2 \\ \frac{3}{2}(\sqrt{3} + \frac{3}{2}) & -\frac{3}{2}(\sqrt{3} + \frac{3}{2}) \end{pmatrix} \approx \begin{pmatrix} -0.40 & 0.40 \\ 0.29 & -0.29 \end{pmatrix} \quad (60b)$$

$$\mathbb{H}_{1,2} = \frac{1}{17} \begin{pmatrix} 9 + 3\sqrt{3} & 8 - 3\sqrt{3} \\ 9 + 3\sqrt{3} & 8 - 3\sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 0.84 & 0.16 \\ 0.84 & 0.16 \end{pmatrix} \quad (61a)$$

$$\mathbb{H}_{2,2} = \frac{1}{272} \begin{pmatrix} -25 + 3\sqrt{3} & 25 - 3\sqrt{3} \\ 3(\sqrt{3} + 3) & -3(\sqrt{3} + 3) \end{pmatrix} \approx \begin{pmatrix} -0.07 & 0.07 \\ 0.05 & -0.05 \end{pmatrix} \quad (61b)$$

$$\mathbb{H}_{1,3} = \frac{1}{34} \begin{pmatrix} 3\sqrt{3} + 9 & 25 - 3\sqrt{3} \\ 3\sqrt{3} + 9 & 25 - 3\sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 0.42 & 0.58 \\ 0.42 & 0.58 \end{pmatrix} \quad (62a)$$

$$\mathbb{H}_{2,3} = \frac{1}{34} \begin{pmatrix} 25 - 3\sqrt{3} & -25 + 3\sqrt{3} \\ -(3\sqrt{3} + 9) & 3\sqrt{3} + 9 \end{pmatrix} \approx \begin{pmatrix} 0.58 & -0.58 \\ -0.42 & 0.42 \end{pmatrix}. \quad (62b)$$

Thus, asymptotically the average number of vehicles associated with each city (whose distribution will repeat daily) at the three moments of one day are

$$\begin{aligned} \begin{pmatrix} a_\infty^{(1)} & b_\infty^{(1)} \end{pmatrix} &= \lim_{r \rightarrow \infty} \begin{pmatrix} a_0 & b_0 \end{pmatrix} \mathbb{M}_1 (\mathbb{M}_2 \mathbb{M}_3 \mathbb{M}_1)^{r-1} = \begin{pmatrix} a_0 & b_0 \end{pmatrix} \mathbb{H}_{1,1} \\ &= (a_0 + b_0) \begin{pmatrix} \frac{3}{34} (7 - \sqrt{3}) & \frac{1}{34} (3\sqrt{3} + 13) \end{pmatrix} \\ &\approx (a_0 + b_0) \begin{pmatrix} 0.46 & 0.54 \end{pmatrix}, \end{aligned} \quad (63)$$

$$\begin{aligned} \begin{pmatrix} a_\infty^{(2)} & b_\infty^{(2)} \end{pmatrix} &= \lim_{k \rightarrow \infty} \begin{pmatrix} a_0 & b_0 \end{pmatrix} \mathbb{M}_1 \mathbb{M}_2 (\mathbb{M}_3 \mathbb{M}_1 \mathbb{M}_2)^r = \begin{pmatrix} a_0 & b_0 \end{pmatrix} \mathbb{H}_{1,2} \\ &= (a_0 + b_0) \begin{pmatrix} \frac{3}{17} (\sqrt{3} + 3) & \frac{1}{17} (8 - 3\sqrt{3}) \end{pmatrix} \\ &\approx (a_0 + b_0) \begin{pmatrix} 0.84 & 0.16 \end{pmatrix}, \end{aligned} \quad (64)$$

$$\begin{aligned} \begin{pmatrix} a_\infty^{(3)} & b_\infty^{(3)} \end{pmatrix} &= \lim_{k \rightarrow \infty} \begin{pmatrix} a_0 & b_0 \end{pmatrix} (\mathbb{M}_1 \mathbb{M}_2 \mathbb{M}_3)^r = \begin{pmatrix} a_0 & b_0 \end{pmatrix} \mathbb{H}_{1,3} \\ &= (a_0 + b_0) \begin{pmatrix} \frac{3}{34} (\sqrt{3} + 3) & \frac{1}{34} (25 - 3\sqrt{3}) \end{pmatrix} \\ &\approx (a_0 + b_0) \begin{pmatrix} 0.42 & 0.58 \end{pmatrix}. \end{aligned} \quad (65)$$

In these last three equations (third lines) the pairs of numbers  $(0.46, 0.54)$ ,  $(0.84, 0.16)$  and  $(0.42, 0.58)$  mean that – for the first pair, for instance, (morning monitoring) –, 46% of all the vehicles,  $a_0 + b_0$ , are those in city  $A$  plus those driving toward  $B$ , while 54% are in  $B$  plus the ones driving toward  $A$ . The same meaning holds for the two other sets, being for noon and evening monitoring respectively. These states change periodically and continuously, as schematized in Figure 8, where the arrows indicate the direction of the flow of vehicles. The asymptotic results do not depend on the initial values  $a_0$  and  $b_0$  individually, but on their sum. As so, some information is lost, but only asymptotically, and the percentages depend only on the parameters of the chosen initial Markov stochastic matrix. Also, the flow of vehicles change cyclically, ruled by the stationary matrices  $\mathbb{H}_{1,i}$  and  $\mathbb{H}_{2,i}$  as displayed in Table IX.

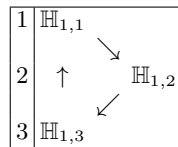


TABLE IX: The stationary traffic flow changes cyclically according to the asymptotic matrices  $\mathbb{H}_{1,i}$ , according to the moments of the day: 1 (morning), 2 (noon) and 3 (evening).



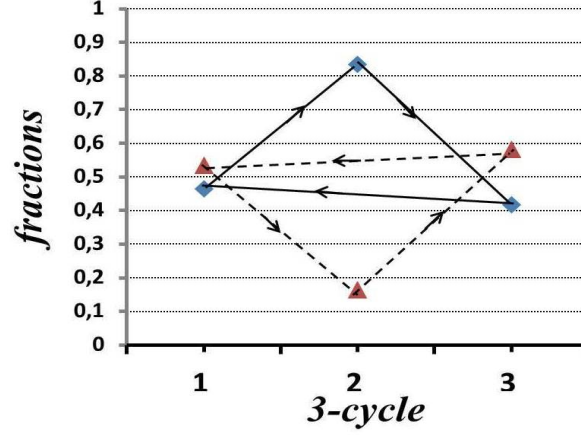


FIG. 8: Three stationary states for each city. The flow of vehicles changes cyclically at each step: 1 (morning), 2 (noon) and 3 (evening). The arrows indicate the directions of the cyclic changes: clockwise (solid lines) for city *A* and counterclockwise (dashed lines) for *B*. The symmetry is due to the choices  $a_1 = a_2 = 1/2$  and  $b_1 = b_2 = -1/2$  for the parameters of the model.

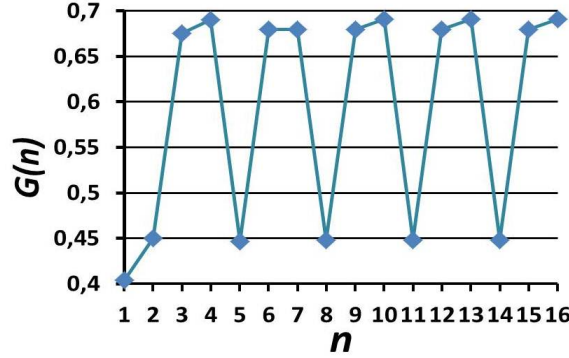


FIG. 9: The global mean entropy for the nonlinear model as function of  $n$ . The solid line only links the points and the 3-cycle signature is the pattern of the periodic repetition.

The global mean entropy also presents asymptotic regular cyclical changes as shown in Fig. 9, which is in line with Fig. 8 and Table IX.

As so, this particular nonlinear model leads to three stationary matrices (three fixed points in the language of dynamical systems) characterizing a 3-cycle continuous change of the traffic flow of vehicles, that we assumed as being three moments of observation each day. This approach can be extended to  $L$  stationary states for the flow of vehicles at  $L$  different moments. The model is scalable by increasing the number of sites and arteries that connect them, and the extraction of numerical results depends only on computational capability. To fit the empirical data we can make the model more kin to the reality by inserting the dynamics into the network of input and output of vehicles (edges in the digraph) in one or more vertices, besides choosing conveniently its parameters.

### VIII. SUMMARY AND CONCLUSIONS

We have here presented some models to treat the traffic flow of vehicles between sites that may be cities, parking lots or car-rental agencies, and arteries (roads) that permit the flow of vehicles between the sites. The aim of modeling is to construct a scheme, based on the  $n$ -step Markov chain matrices, that simulate the evolution that could fit the empirical data. By doing a convenient decomposition of the  $n$ -step Markov stochastic matrix in several modes we have separated the stationary state from the transient ones, for which we have also defined characteristic decay times. We have initially treated a network without source and sink for the circulation of vehicles and verified that it takes very few steps to attain the stationary state; we also described the evolution by a measure: the global mean entropy. The

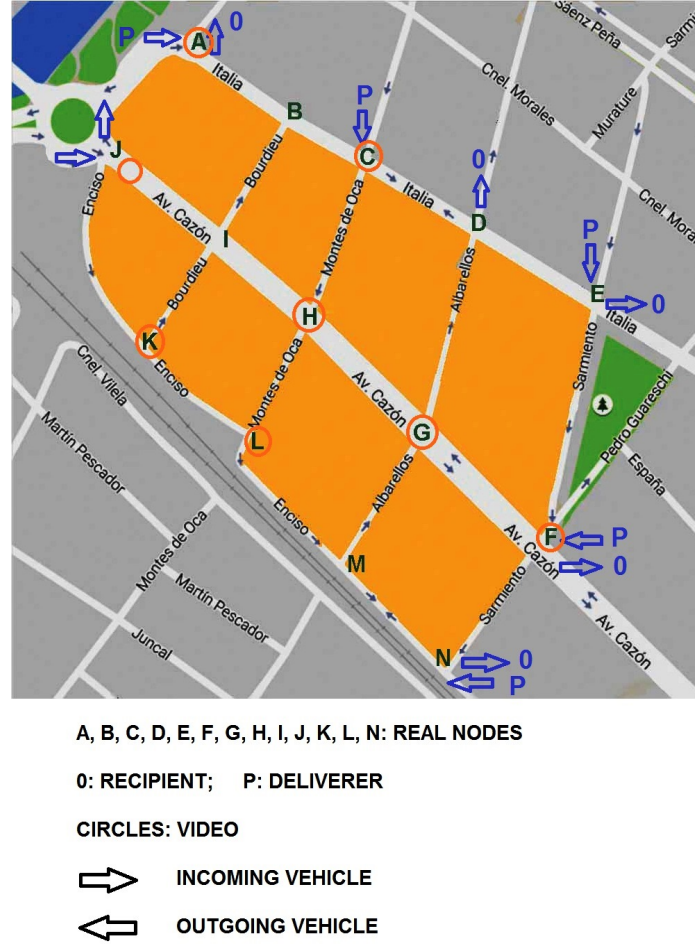


FIG. 10: Downtown of the county of Tigre. The chosen region consists of 14 intersections, with 8 cameras installed to collect data of the traffic flow.

second model consists in implementing a source and a sink within the former, namely, two additional sites (vertices in terms of digraphs) and routes (edges), one site containing a certain number of new vehicles that are injected daily into the network at an established rate, and the other site is where old and crashed vehicles are recalled from the network, also at an established rate. According to chosen numbers for the entries of the matrices, the main differences that resulted from the former model are: (1) very long decay times to attain the stationary state, where all the vehicles will agglomerate in the depot site; (2) the evolution of the global mean entropy increases, it attains a maximum and then decreases, going to zero, since all the vehicles go eventually to the sink. This model could help urban planners to establish what could be the ideal injection of vehicles within a network in order to avoid an excess that could surpass the capacity of the existing roads and parking lots.

The third model goes beyond the linear models (a single stationary state), it is nonlinear because the entries of the stochastic matrix contain a dependence on  $n$ . In order to illustrate the model we have considered two sites and two arteries connecting them. The time dependence we adopted relies on the use of periodic functions. Compared to the linear models, the changes are quite remarkable: instead of a single stationary state, multistationary states are possible and each mode has its own relaxation time although the PF eigenvalue ( $\lambda_{PF} = 1$ ) is always present, even for the  $n$ -step evolution. We worked out an exact soluble model in order to illustrate the method and chose a specific sinusoidal dependence on  $n$ , causing the occurrence of three stationary states, that permitted to predict the possibility that the traffic flow may change cyclically. We consider that this model is more realistic to describe the possible variation of the traffic flow at different moments of the day beyond the transient regimes. These features can be extended to  $L$  stationary states. Finally, we recall that all three models are scalable, i.e., depending on the computational resources the network can be extended to a large number of vertices and edges. This study is a preliminary theoretical approach of observation and recording that is being done in the county of Tigre, a province of Buenos Aires, Argentina, see Figure 10.

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